

PARTIAL ACTIONS OF C^* -QUANTUM GROUPS I: RESTRICTION AND GLOBALIZATION

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ABSTRACT. Partial actions of groups on C^* -algebras and the closely related actions and coactions of Hopf algebras received much attention over the last decades. They arise naturally as restrictions of their global counterparts to non-invariant subalgebras, and the ambient enveloping global (co)actions have proven useful for the study of associated crossed products. In this article, we introduce the partial coactions of C^* -bialgebras, focussing on C^* -quantum, and prove existence of an enveloping global coaction under mild technical assumptions. The construction of the latter provides a left adjoint to the forgetful functor from coactions to partial coactions. We also show that partial coactions of the function algebra of a discrete group correspond to partial actions on direct summands of a C^* -algebra, and relate partial coactions of a compact or its dual discrete C^* -quantum group to partial coactions or partial actions of the dense Hopf subalgebra.

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1. INTRODUCTION

Partial actions of groups on spaces and on C^* -algebras were gradually introduced in [15], [16], [21], and the study of associated crossed products has shed new lights on the inner structure of many interesting C^* -algebras; see [17] for a comprehensive introduction and an overview. In the purely algebraic setting, the corresponding notion of a partial action or a partial coaction of a Hopf algebra on an algebra was introduced in [12].

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Naturally, such partial (co)actions arise by restricting global (co)actions to non-invariant subspaces or ideals, and in these cases, all the tools that are available for the study of global situation can be applied to the study of the partial one. Therefore, it is highly desirable to know, given a partial group action or a partial Hopf algebra (co)action, whether it can be identified with some restriction of a global one, whether there exists a minimal global one — called a *globalization* — and whether the latter, if it exists, can be constructed explicitly. For partial actions of groups on locally compact Hausdorff spaces, such a globalization can always be constructed, but the underlying space need no longer be Hausdorff [1], [2]. As a consequence, partial actions of groups on C^* -algebras can not always be identified with the restriction of a global action [2]. In the purely algebraic setting, partial (co)actions of Hopf algebras always have a globalization [4], [5]; see also [3], [6], [14].

In this article, we introduce partial coactions of C^* -bialgebras, in particular, of C^* -quantum groups, on C^* -algebras, and relate them to the partial (co)actions discussed above. In case of the function algebra of a discrete group, partial coactions correspond to partial actions of groups where for every group element, the associated domain of definition is a direct summand of the total C^* -algebra, and these are precisely the partial actions for which existence of a globalization can be proven. If the C^* -bialgebra is a discrete C^* -quantum group, then every partial coaction gives rise to a partial action of the Hopf algebra of matrix coefficients of the dual compact quantum group. Finally, in case of a compact C^* -quantum group, partial coactions restrict, under a natural condition, to partial coactions of the Hopf algebra of matrix coefficients on a dense subalgebra.

The main result of this article is the existence and a construction of a globalization under mild assumptions. We follow the approach for coactions of Hopf algebras [5], but face new technical difficulties. As a consequence, we need to assume that the C^* -algebra of the quantum group under consideration has the slice map property, which follows, for example, from nuclearity [29], and is automatic if the quantum group is discrete. If this condition holds, we obtain an enveloping coaction in a functorial way.

Categorically, the main result can be summarized as follows. If we regard coactions as special cases of partial coactions, we obtain a forgetful functor from the former to the latter, and if we choose the classes of morphisms in both categories with some care, the construction of the enveloping coaction furnishes a left adjoint:

Theorem (see 8.15). *Let (A, Δ) be a C^* -quantum group, where A has the slice map property. Then the forgetful functor from injective, weakly continuous coactions of (A, Δ) to injective, weakly continuous, regular partial coactions has a left adjoint. The counit of this adjunction is the identity.*

Parts of the results in this article were obtained in the Master's theses of the first and the second author. In following articles, we plan to study crossed products for partial coactions, and partial corepresentations of C^* -bialgebras.

The article is organized as follows. In Section 2, we recall background on C^* -quantum groups, strict $*$ -homomorphisms and the slice map property. In Section 3, we introduce partial coactions of C^* -bialgebras and discuss a few desirable properties like weak and strong continuity. In Section 4, we show that partial actions of a discrete group Γ on a C^* -algebra correspond to counital coactions of the function algebra $C_0(\Gamma)$ if and only if the domains of definition are direct summands of the C^* -algebra. In Section 5, we relate

partial coactions of compact and of discrete C^* -quantum groups to coactions and actions of the Hopf algebra of matrix elements of the compact quantum group. In Section 6, we show how partial coactions arise from global ones by restriction, and discuss the closely related notion of weak or strong morphisms between partial coactions. In Section 7, we consider the situation where a partial coaction can be identified with the restriction of a global coaction, and study a few preliminary properties of such identifications. Finally, in Section 8, we prove the main result stated above.

2. PRELIMINARIES

Let us fix some notation and recall some background.

Conventions and notation. Given a locally compact Hausdorff space X , we denote by $C_b(X)$ and $C_0(X)$ the C^* -algebra of continuous functions that are bounded or vanish at infinity, respectively.

For a subset F of a normed space E , we denote by $[F] \subseteq E$ its closed linear span.

Given a C^* -algebra A , we denote by A' the space of bounded linear functionals on A , by $M(A)$ the multiplier algebra and by $1_A \in M(A)$ the unit of $M(A)$.

Given a Hilbert space K , we denote by 1_K the identity on H .

Let A and B be C^* -algebras. A $*$ -homomorphism $\varphi: A \rightarrow M(B)$ is called *nondegenerate* if $[\varphi(A)B] = B$. Each nondegenerate $*$ -homomorphism $\varphi: A \rightarrow M(B)$ extends uniquely to a unital $*$ -homomorphism from $M(A)$ to $M(B)$, which we denote by ϕ again. By a *representation* of a C^* -algebra A on a Hilbert space H we mean a $*$ -homomorphism $\pi: A \rightarrow \mathcal{B}(H)$. All tensor products of C^* -algebras will be minimal ones.

We write σ for the tensor flip isomorphism $A \otimes B \rightarrow B \otimes A$, $a \otimes b \mapsto b \otimes a$.

C^* -bialgebras and C^* -quantum groups. A C^* -bialgebra is a C^* -algebra A with a non-degenerate $*$ -homomorphism $\Delta: A \rightarrow M(A \otimes A)$, called the *comultiplication*, that is coassociative in the sense that $(\Delta \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \Delta) \circ \Delta$. It satisfies the *cancellation conditions* if

$$(2.1) \quad [\Delta(A)(1_A \otimes A)] = A \otimes A = [(A \otimes 1_A)\Delta(A)].$$

Given a C^* -bialgebra (A, Δ) , the dual space A' is an algebra with respect to the convolution product defined by $v\omega := (v \otimes \omega) \circ \Delta$.

A *counit* for a C^* -bialgebra (A, Δ) is a character ε on A satisfying $(\varepsilon \otimes \text{id}_A) \circ \Delta = \text{id}_A = (\text{id}_A \otimes \varepsilon) \circ \Delta$. If it exists, such a counit is a unit in the algebra A' and thus unique.

A *morphism* of C^* -bialgebras (A, Δ_A) and (B, Δ_B) is a non-degenerate $*$ -homomorphism $f: A \rightarrow M(B)$ satisfying $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$.

A C^* -quantum group is a C^* -bialgebra that arises from a well-behaved multiplicative unitary as follows [26, 27, 30]. Suppose that H is a Hilbert space and that $W \in \mathcal{B}(H \otimes H)$ is a multiplicative unitary [8] that is manageable or modular [30, 27]. Then the spaces

$$A := [(\omega \otimes \text{id}_H)W : \omega \in \mathcal{B}(H)_*] \quad \text{and} \quad \hat{A} := [(\text{id}_H \otimes \omega)W : \omega \in \mathcal{B}(H)_*]$$

are separable, nondegenerate C^* -subalgebras of $\mathcal{B}(H)$, the unitary W is a multiplier of $\hat{A} \otimes A \subseteq \mathcal{B}(H \otimes H)$, and the formulas

$$(2.2) \quad \Delta(a) = W(a \otimes 1_H)W^*, \quad \hat{\Delta}(\hat{a}) = \sigma(W^*(1_H \otimes \hat{a})W)$$

define comultiplications on A and \hat{A} , respectively, such that (A, Δ) and $(\hat{A}, \hat{\Delta})$ become C^* -bialgebras. A C^* -bialgebra (A, Δ) is a C^* -quantum group if it arises from a modular multiplicative unitary W as above.

Let (A, Δ) be a C^* -quantum group arising from a unitary W as above. Denote by Σ the flip on $H \otimes H$. Then also the *dual* $\widehat{W} := \Sigma W^* \Sigma$ of W is a modular or manageable multiplicative unitary and the associated C^* -quantum group is $(\hat{A}, \hat{\Delta})$. The latter only depends on (A, Δ) and not on the choice of W , and is called the *dual* of (A, Δ) . The images of W and \widehat{W} in $M(\hat{A} \otimes A)$ or $M(A \otimes \hat{A})$, respectively, do not depend on the choice of W but only on (A, Δ) . We call them the *reduced bicharacters* of (A, Δ) and $(\hat{A}, \hat{\Delta})$ and denote them by W^A and \widehat{W}^A , respectively. We will need an *anti-Heisenberg pair* for (A, Δ) , which consists of non-degenerate, faithful representations π of A and $\hat{\pi}$ of \hat{A} on a Hilbert space K such that the unitary

$$(2.3) \quad V := (\text{id}_A \otimes \hat{\pi})(\widehat{W}^A) \in M(A \otimes \hat{\pi}(\hat{A})),$$

regarded as an element of $M(A \otimes \mathcal{K}(K))$, satisfies

$$(2.4) \quad V(1_A \otimes \pi(a))V^* = (\text{id}_A \otimes \pi)\Delta(a) \text{ for all } a \in A;$$

see [22, §3] and [25, §3.1].

We call a C^* -quantum group (A, Δ) *weakly regular* [24, Definition 5.37] if its reduced bicharacter satisfies $[(\hat{A} \otimes 1_A)W^A(1_{\hat{A}} \otimes A)] = \hat{A} \otimes A$ in $M(\hat{A} \otimes A)$. This is equivalent to the condition $[(1_{\hat{A}} \otimes A)W^A(\hat{A} \otimes 1_A)] = \hat{A} \otimes A$, see [24, Proof of Corollary 5.39]. For the unitary (2.3), this translates into

$$(2.5) \quad [(1_A \otimes \hat{\pi}(\hat{A}))V(A \otimes 1_{\hat{\pi}(\hat{A})})] = A \otimes \hat{\pi}(\hat{A}) \text{ in } M(A \otimes \hat{\pi}(\hat{A})).$$

Every locally compact quantum group or, more precisely, every reduced C^* -algebraic quantum group in the sense of Kustermans and Vaes [18], is a C^* -quantum group. If the former is regular in the sense of [9, §5(b)], then it is also weakly regular in the sense above, see [8, Proposition 3.6].

A *compact C^* -quantum group* is, by definition, a unital C^* -bialgebra $\mathbb{G} = (A, \Delta)$ that satisfies the cancellation conditions, and is indeed a weakly regular C^* -quantum group [31]. Associated to such a compact quantum group is a rigid C^* -tensor category of unitary finite-dimensional corepresentations [23]. We denote by $\text{Irr}(\mathbb{G})$ the equivalence classes of irreducible corepresentations. Their matrix elements span a dense Hopf subalgebra $\mathcal{O}(\mathbb{G})$. The dual $(\hat{A}, \hat{\Delta})$ is called a *discrete C^* -quantum group*, and the underlying C^* -algebra \hat{A} is a direct sum of matrix algebras, indexed by $\text{Irr}(\mathbb{G})$.

Strict $*$ -homomorphisms of C^* -algebras. Recall from [19, §5, Corollary 5.7] that a $*$ -homomorphism $\pi: B \rightarrow M(C)$ is *strict* if it is strictly continuous on the unit ball, and that in that case, it extends to a $*$ -homomorphism $M(B) \rightarrow M(C)$ that is strictly continuous on the unit ball. We denote this extension by π again. Using this extension, we define the composition of strict $*$ -homomorphisms, which evidently is strict again. Hence, C^* -algebras with strict $*$ -homomorphisms form a category.

Recall that a *corner* of a C^* -algebra B is a C^* -subalgebra of the form pBp for some projection $p \in M(B)$.

Strict $*$ -homomorphisms are just non-degenerate $*$ -homomorphisms in the usual sense from the domain to a corner of the target. Indeed, if $\pi: B \rightarrow M(C)$ is a strict $*$ -homomorphism, then $p := \pi(1_B) \in M(C)$ is a projection, $pCp \subseteq C$ is a corner, and the co-restriction $\pi: B \rightarrow M(pCp)$ is non-degenerate. Conversely, given a corner $C_0 \subseteq C$ and a non-degenerate $*$ -homomorphism $\pi: B \rightarrow M(C_0)$, we get a strict extension $M(B) \rightarrow M(C_0)$, a natural strict map $M(C_0) \rightarrow M(C)$ [10, II.7.3.14], and the composition is a strict $*$ -homomorphism.

This description of strict $*$ -homomorphisms immediately implies that the minimal tensor product of partial morphisms is a partial morphism again, and that an embedding of C^* -algebras $B \hookrightarrow C$ is a strict $*$ -homomorphism if and only if B is a non-degenerate C^* -subalgebra of a corner of C . We shall call such embeddings *strict*.

In the commutative case, partial morphisms correspond to partially defined continuous maps with clopen domain of definition. Indeed, let X and Y be locally compact Hausdorff spaces. Then every continuous map F from a clopen subset $D \subseteq Y$ to X induces a strict $*$ -homomorphism $F^*: C_0(X) \rightarrow M(C_0(Y)) = C_b(Y)$ defined by

$$(F^*(f))(y) = 0 \text{ if } y \notin D, \quad (F^*(f))(y) = f(F(y)) \text{ if } y \in D.$$

Conversely, if $\pi: C_0(X) \rightarrow M(C_0(Y))$ is a strict $*$ -homomorphism, then $\pi(1_X)$ is the characteristic function of a clopen subset $D \subseteq Y$ and the corestriction $\pi: C_0(X) \rightarrow M(C_0(D))$ is the pull-back along a continuous function $F: D \rightarrow X$.

2.1. The slice map property. In sections 7 and 8, we need the following property. A C^* -algebra A has the *slice map property* if for every C^* -algebra B and every C^* -subalgebra $C \subseteq B$, every $x \in B \otimes A$ satisfying $(\text{id} \otimes \omega)(x) \in C$ for all $\omega \in A'$ lies in $C \otimes A$ [29]. This property holds if A is nuclear, or, more generally, if A has the completely bounded approximation property or the strong operator approximation property; see [28] for a survey. In particular, this condition holds whenever (A, Δ) is a discrete quantum group, or, more generally, whenever (A, Δ) is a reduced C^* -algebraic quantum group whose dual is amenable [11, Theorem 3.3].

3. PARTIAL COACTIONS OF C^* -BIALGEBRAS

The definition of a partial coaction given for Hopf algebras in [13] carries over to C^* -bialgebras as follows.

3.1. Definition. A partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra C is a strict $*$ -homomorphism $\delta: C \rightarrow M(C \otimes A)$ satisfying the following conditions:

- (1) $\delta(C)(1_C \otimes A) \subseteq C \otimes A$;
- (2) δ is partially coassociative in the sense that

$$(3.1) \quad (\delta \otimes \text{id}_A)\delta(c) = (\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta(c)$$

for all $c \in C$, or, equivalently, the following diagram commutes:

$$(3.2) \quad \begin{array}{ccc} C & \xrightarrow{\delta} & M(C \otimes A) \\ \delta \downarrow & & \downarrow \delta \otimes \text{id} \\ M(C \otimes A) & \xrightarrow{(\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta} & M(C \otimes A \otimes A) \end{array}$$

Let δ be a partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra C . For every functional $\omega \in A'$ and every multiplier $T \in M(C)$, we define a multiplier

$$\omega \triangleright T := (\text{id}_C \otimes \omega)\delta(T) \in M(C),$$

where we use the fact that we can write $\omega = av$ or $\omega = v'a'$ with $a, a' \in A$ and $v, v' \in A'$ by Cohen's factorization theorem.

Let $c \in C$ and $\omega \in A'$. Then conditions (1) and (2) in Definition 3.1 imply $\omega \triangleright c \in C$ and

$$(3.3) \quad \delta(\omega \triangleright c) = (\text{id}_C \otimes \text{id}_A \otimes \omega)(\delta \otimes \text{id}_A)\delta(c) = \delta(1_C)(\text{id}_C \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta)\delta(c).$$

In particular, for every character $\chi \in A'$,

$$(3.4) \quad \chi \triangleright (\omega \triangleright c) = (\chi \triangleright 1_C)(\text{id}_C \otimes (\chi \otimes \omega)\Delta)\delta(c) = (\chi \triangleright 1_C)(\chi \omega \triangleright c).$$

The following conditions on a partial coaction are straightforward generalizations of the corresponding conditions on coactions, and will play an equally important role:

3.2. Definition. We call a partial coaction δ of a C^* -bialgebra (A, Δ) on a C^* -algebra C

- (strongly) continuous if $[\delta(C)(1_C \otimes A)] = [\delta(1_C)(C \otimes A)]$;
- weakly continuous if $[A' \triangleright C] = C$;
- counital if (A, Δ) has a counit ε and $(\text{id}_C \otimes \varepsilon) \circ \delta = \text{id}$.

3.3. Remark. If δ is a partial coaction as above and $X \subseteq A'$ is a subset that separates the points of A , then a standard application of the Hahn-Banach theorem shows that $[X \triangleright C] = [A' \triangleright C]$.

Every counital partial coaction evidently is weakly continuous.

3.4. Lemma. Let δ be a strongly continuous partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra C . Then:

- (1) δ is weakly continuous if and only if $[(A' \triangleright 1_C)C] = C$;
- (2) δ is counital if and only if (A, Δ) has a counit ε and $\varepsilon \triangleright 1_C = 1_C$.

Proof. (1) By assumption, the closed linear span of all elements of the form $a\omega \triangleright c = (\text{id}_C \otimes \omega)(\delta(c)(1_C \otimes a))$, where $\omega \in A'$, $a \in A$ and $c \in C$, is equal to the closed linear span of all elements of the form $(\text{id}_C \otimes \omega)(\delta(1_C)(c \otimes a)) = (a\omega \triangleright 1_C)c$. Now, use Cohen's factorization theorem.

(2) If $\varepsilon \triangleright 1_C = 1_C$, then elements of the form $a\varepsilon \triangleright c$, where $a \in A$ and $c \in C$, are linearly dense in C , and for every $\omega \in A'$ and $c \in C$, (3.4) implies $\varepsilon \triangleright (\omega \triangleright c) = 1_C \cdot (\omega \triangleright c)$. \square

For regular reduced C^* -algebraic quantum groups, weakly continuous coactions are automatically strongly continuous [9, Proposition 5.8]. More generally, we show:

3.5. Proposition. Let (A, Δ) be a weakly regular C^* -quantum group. Then every weakly continuous partial coaction of (A, Δ) is strongly continuous.

Proof. We proceed similarly as in the proof of [9, Proposition 5.8], and use an anti-Heiseberg pair $(\pi, \hat{\pi})$ for (A, Δ) on some Hilbert space K and the unitary V in (2.3).

Let δ be a weakly continuous partial coaction of (A, Δ) on a C^* -algebra C . By (3.3) and Remark 3.3,

$$\begin{aligned} [\delta(C)(1_C \otimes A)] &= [\delta(\omega \circ \pi \triangleright C)(1_C \otimes A) : \omega \in \mathcal{B}(K)_*] \\ &= [\delta(1_C) \cdot (\text{id}_C \otimes \text{id}_A \otimes \omega \circ \pi)((\text{id}_C \otimes \Delta)(\delta(C))) \cdot (1_C \otimes A) : \omega \in \mathcal{B}(K)_*]. \end{aligned}$$

To shorten the notation, let $\delta_\pi := (\text{id}_C \otimes \pi) \circ \delta$. We use the relations (2.4), (2.5) and $[\hat{\pi}(\hat{A})\mathcal{B}(K)_*] = \mathcal{B}(K)_*$, and find

$$\begin{aligned} &[(\text{id}_C \otimes \text{id}_A \otimes \omega \circ \pi)((\text{id}_C \otimes \Delta)(\delta(C))(1_C \otimes A \otimes 1_A)) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \omega)(V_{23}\delta_\pi(C)_{13}V_{23}^*(A \otimes \hat{\pi}(\hat{A}))_{23}) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \omega)(V_{23}\delta_\pi(C)_{13}(A \otimes \hat{\pi}(\hat{A}))_{23}) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \omega)((1_A \otimes \hat{\pi}(\hat{A}))_{23}V_{23}(A \otimes 1_K)_{23}\delta_\pi(C)_{13}) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \omega)((A \otimes \hat{\pi}(\hat{A}))_{23}\delta_\pi(C)_{13}) : \omega \in \mathcal{B}(K)_*] \\ &= [A' \triangleright C] \otimes A, \end{aligned}$$

whence $[\delta(C)(1_C \otimes A)] = [\delta(1_C)(C \otimes A)]$. \square

The first examples of partial coactions that we consider are partial coactions on \mathbb{C} .

3.6. Lemma. *Partial coactions of a C^* -bialgebra (A, Δ) on \mathbb{C} correspond bijectively with projections $p \in M(A)$ satisfying*

$$(3.5) \quad (p \otimes 1_A)\Delta(p) = p \otimes p.$$

Proof. Projections $p \in M(A)$ correspond to strict $*$ -homomorphisms $\delta: \mathbb{C} \rightarrow M(\mathbb{C} \otimes A) \cong M(A)$ via $p = \delta(1)$, and under this correspondence, $(\delta \otimes \text{id}_A)\delta(\lambda) = \lambda \otimes p \otimes p$ and $(\delta(1) \otimes 1_A)(\text{id}_\mathbb{C} \otimes \Delta)(\delta(\lambda)) = \lambda \otimes (p \otimes 1_A)\Delta(p)$. \square

3.7. Example. Let G be a locally compact group.

- (1) Consider the C^* -bialgebra $(C_0(G), \Delta)$. A projection $p \in M(C_0(G))$ is just the characteristic function of a clopen subset $H \subseteq G$, and satisfies (3.5) if and only if $p(g)p(gg') = p(g)p(g')$ for all $g, g' \in G$, that is, if and only if $H \subseteq G$ is a subgroup. Thus, partial coactions of $(C_0(G), \Delta)$ on \mathbb{C} correspond to open subgroups of G .
- (2) Consider the reduced group C^* -bialgebra $(C_r^*(G), \Delta)$. For every finite normal subgroup $N \subseteq G$, the sum $p = \sum_{g \in N} \lambda_g$ is a central projection in $M(C_r^*(G))$ satisfying (3.5), where λ_g denotes the left translation by $g \in G$.

Every central projection satisfying (3.5) gives rise to a quotient C^* -bialgebra (A_p, Δ_p) of (A, Δ) whose coactions can be regarded as partial coactions of (A, Δ) :

3.8. Lemma. *Suppose that (A, Δ) is a C^* -bialgebra with a central projection $p \in M(A)$ satisfying (3.5). Let $A_p = pA$ and define $\Delta_p: A_p \rightarrow M(A_p \otimes A_p)$ by $a \mapsto (p \otimes p)\Delta(a)$. Then (A_p, Δ_p) is a C^* -bialgebra, the map $A \rightarrow A_p$, $a \mapsto pa$, is a morphism of C^* -bialgebras, and every coaction of (A_p, Δ_p) can be regarded as a partial coaction of (A, Δ) .*

Proof. All of these assertions are easily verified, for example, if δ is a coaction of (A_p, Δ_p) on a C^* -algebra C , then for all $c \in C$,

$$\begin{aligned} (\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta(c) &= (1_C \otimes p \otimes 1_A)(1_C \otimes \Delta)((1_C \otimes p)\delta(c)) \\ &= (1_C \otimes p \otimes p)(\text{id}_C \otimes \Delta)\delta(c) = (1_C \otimes \Delta_p)\delta(c) = (\text{id}_C \otimes \delta)\delta(c). \quad \square \end{aligned}$$

3.9. Example. Let $\mathbb{G} = (A, \Delta)$ be a discrete quantum group, so that A is a c_0 -sum of matrix algebras indexed by $\text{Irr}(\hat{\mathbb{G}})$. Consider a central projection $p \in M(A)$ supported on $\mathcal{J} \subseteq \text{Irr}(\hat{\mathbb{G}})$. Then $(p \otimes 1)\Delta(p) = p \otimes p$ if and only if the following condition holds:

$$(3.6) \quad \text{If } \alpha \in \mathcal{J}, \beta, \gamma \in \text{Irr}(\hat{\mathbb{G}}) \text{ and } \alpha \otimes \beta \text{ contains } \gamma, \text{ then } \beta \in \mathcal{J} \text{ if and only if } \gamma \in \mathcal{J}.$$

If (A_p, Δ_p) is a discrete quantum subgroup of (A, Δ) , then \mathcal{J} is closed under taking duals and summands of tensor products, and then Frobenius duality implies (3.6). Conversely, suppose that (3.6) holds. Taking $\gamma = \alpha$, we see that \mathcal{J} contains the trivial representation, and taking this for γ , we see that \mathcal{J} contains the dual of α . Thus, finite sums of representations in \mathcal{J} form a rigid tensor subcategory, and (A_p, Δ_p) is a discrete quantum subgroup of (A, Δ) .

4. THE RELATION TO PARTIAL ACTIONS OF GROUPS

We now relate partial actions of a (discrete) group Γ to counital partial coactions of the C^* -bialgebra $C_0(\Gamma)$. Recall that a *partial action* of Γ on a C^* -algebra C is a family $(D_g)_{g \in \Gamma}$ of closed ideals of C together with a family $(\theta_g)_{g \in \Gamma}$ of isomorphisms $\theta_g: D_{g^{-1}} \rightarrow D_g$ such that

- (G1) $D_e = C$ and $\theta_e = \text{id}_C$, where $e \in \Gamma$ denotes the unit,
- (G2) $\theta_{g^{-1}}\theta_g\theta_h = \theta_{g^{-1}}\theta_{gh}$ and $\theta_g\theta_h\theta_{h^{-1}} = \theta_{gh}\theta_{h^{-1}}$ for all $g, h \in \Gamma$ as partially defined maps;

see [17, 21]. We show that partial coactions of $C_0(\Gamma)$ correspond to partial actions of Γ as above, where each ideal D_g is a direct summand, and adopt the following terminology:

4.1. Definition. A disconnected partial action of Γ on a C^* -algebra C is given by a family $(p_g)_{g \in \Gamma}$ of central projections in $M(C)$ and a family $(\theta_g)_{g \in \Gamma}$ of isomorphisms $\theta_g: p_{g^{-1}}C \rightarrow p_gC$ such that $((p_gC)_{g \in \Gamma}, (\theta_g)_{g \in \Gamma},)$ is a partial action.

We denote by $C_b(X; C)$ the C^* -algebra of norm-bounded C -valued functions on Γ , and identify this C^* -algebra with a subalgebra of $M(C \otimes C_0(\Gamma))$ in the canonical way. For each $g \in \Gamma$, we denote by $\text{ev}_g \in C_0(\Gamma)'$ the evaluation at g .

4.2. Proposition. Let Γ be a group and let C be a C^* -algebra.

- (1) Let δ be a counital partial coaction of $C_0(\Gamma)$ on C . Then the projections

$$p_g := \text{ev}_g \triangleright 1_C$$

are central and the maps $\theta_g: p_{g^{-1}}C \rightarrow p_gC$ given by

$$\theta_g(c) := \text{ev}_g \triangleright c$$

form a disconnected partial action of Γ on C .

(2) Let $((p_g)_{g \in \Gamma}, (\theta_g)_{g \in \Gamma})$ be a disconnected partial action of Γ on C . Then the map

$$\delta: C \rightarrow C_b(\Gamma; C) \hookrightarrow M(C \otimes C_0(\Gamma))$$

defined by

$$(\delta(c))(g) := \theta_g(p_{g^{-1}}c) \quad (c \in C, g \in \Gamma)$$

is a counital partial coaction of $C_0(\Gamma)$ on C .

Proof. (1) For each $g \in \Gamma$, the map $\Theta_g: C \rightarrow C$ given by $c \mapsto \text{ev}_g \triangleright c$ is a strict endomorphism. Since δ is counital, Θ_e is the identity on C . Let $g, h \in \Gamma$. Then by (3.4),

$$(4.1) \quad \Theta_g(\Theta_h(c)) = (\text{ev}_g \triangleright 1_C)(\text{ev}_g \text{ev}_h \triangleright c) = p_g \Theta_{gh}(c),$$

in particular,

$$(4.2) \quad \Theta_g(p_h) = p_g p_{gh}, \quad \Theta_g(\Theta_{g^{-1}}(c)) = p_g c, \quad \Theta_{g^{-1}}(\Theta_g(c)) = p_{g^{-1}}c.$$

Since $\Theta_g \circ \Theta_{g^{-1}}$ is a $*$ -homomorphism, the second equation implies $p_g c = c p_g$ for all $c \in C$, that is, p_g is central and $D_g := p_g C$ is a direct summand of C . The second and third equations imply that Θ_g and $\Theta_{g^{-1}}$ restrict to mutually inverse isomorphisms

$$D_{g^{-1}} \xrightleftharpoons[\theta_{g^{-1}}]{\theta_g} D_g.$$

It remains to show that $\theta_{g^{-1}}\theta_{gh} = \theta_{g^{-1}}\theta_g\theta_h$. But the relations (4.1) and (4.2) imply that

$$(\Theta_{g^{-1}} \circ \Theta_{gh})(c) = p_{g^{-1}}\Theta_{gh}(c) = (\Theta_{g^{-1}} \circ \Theta_g \circ \Theta_h)(c)$$

for all $c \in C$, and that the compositions $\theta_{g^{-1}}\theta_{gh}$ and $\theta_{g^{-1}}\theta_g\theta_h$ have the domain

$$\Theta_{h^{-1}g^{-1}}(p_g)C = p_{h^{-1}g^{-1}}p_{h^{-1}}C = \Theta_{h^{-1}}(p_{g^{-1}})C.$$

(2) For each $g \in \Gamma$, denote by $\delta_g \in C_0(\Gamma)$ the characteristic function of $\{g\} \subset \Gamma$. Then

$$\delta(c)(1_C \otimes \delta_g) = \theta_g(p_g c) \otimes \delta_g \quad (g \in \Gamma, c \in C).$$

We conclude that $\delta(C)(1_C \otimes C_0(\Gamma))$ is contained in $C \otimes C_0(\Gamma)$, and that δ co-restricts to a non-degenerate $*$ -homomorphism from C to $q(C \otimes C_0(\Gamma))$, where $q = \sum_{g \in \Gamma} p_g \otimes \delta_g$, so that δ is strict. To verify that δ is partially coassociative, it suffices to check that for all $g, h \in \Gamma$ and $c \in C$, the element

$$(\text{id}_C \otimes \text{ev}_g \otimes \text{ev}_h)(\delta \otimes \text{id}_A)\delta(c) = \theta_g(p_{g^{-1}}\theta_h(p_{h^{-1}}c))$$

is equal to the element

$$(\text{id}_C \otimes \text{ev}_g \otimes \text{ev}_h)((\delta(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)\delta(c)) = \theta_g(p_{g^{-1}})\theta_{gh}(p_{h^{-1}g^{-1}}c),$$

and this follows easily from the definition of a partial action. \square

The following example shows that the correspondence between partial coactions of $C_0(\Gamma)$ and partial actions of Γ does not easily extend from groups to inverse semigroups.

4.3. Example. Denote by Γ the inverse semigroup consisting of the 2×2 -matrices

$$0, \quad v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad v^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad vv^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v^*v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with matrix multiplication as composition. For $x \in \Gamma$, define $\delta_x \in C(\Gamma)$ by $y \mapsto \delta_{x,y}$. Then the $*$ -homomorphism

$$\delta: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes C(\Gamma), \quad (\alpha, \beta) \mapsto (\alpha, 0) \otimes \delta_{v^*v} + (0, \alpha) \otimes \delta_v,$$

is a partial coaction. Indeed, for all $\alpha, \beta \in \mathbb{C}$,

$$(\delta \otimes \text{id}_{C(\Gamma)})\delta((\alpha, \beta)) = (\alpha, 0) \otimes \delta_{v^*v} \otimes \delta_{v^*v} + (0, \alpha) \otimes \delta_v \otimes \delta_{v^*v}$$

is equal to the product of

$$\delta((1, 0)) \otimes 1_{C(\Gamma)} = (1, 0) \otimes \delta_{v^*v} \otimes 1_{C(\Gamma)} + (0, 1) \otimes \delta_v \otimes 1_{C(\Gamma)}$$

with

$$\begin{aligned} (\text{id}_{\mathbb{C}^2} \otimes \Delta)\delta((\alpha, \beta)) &= (\alpha, 0) \otimes (\delta_{v^*v} \otimes \delta_{v^*} + \delta_{vv^*} \otimes \delta_{vv^*}) \\ &\quad + (0, \alpha) \otimes (\delta_v \otimes \delta_{v^*v} + \delta_{vv^*} \otimes \delta_v). \end{aligned}$$

But the maps $\Theta_w := (\text{id} \otimes \text{ev}_w) \circ \delta$, where $w \in \Gamma$, are given by

$$\Theta_0 = \Theta_{v^*} = \Theta_{vv^*} = 0, \quad \Theta_v((\alpha, \beta)) = (0, \alpha), \quad \Theta_{v^*v}((\alpha, \beta)) = (\alpha, 0);$$

in particular, $\Theta_v \Theta_{v^*} \Theta_v = 0$ and $\Theta_v \Theta_{v^*v} = \Theta_v$.

5. PARTIAL COACTIONS OF DISCRETE AND OF COMPACT C^* -QUANTUM GROUPS

Let $\mathbb{G} = (A, \Delta)$ be a compact C^* -quantum group and denote by $\mathcal{O}(\mathbb{G}) \subseteq A$ the dense Hopf subalgebra of matrix elements of finite-dimensional corepresentations. We now relate partial (co)actions of \mathbb{G} and of the discrete dual $\hat{\mathbb{G}}$ to partial coactions and partial actions of the Hopf algebra $\mathcal{O}(\mathbb{G})$, respectively. Note that (A, Δ) and $(\hat{A}, \hat{\Delta})$ are weakly regular, so that weakly continuous partial coactions are automatically a strongly continuous by Proposition 3.5.

Recall that a *partial action* of a Hopf algebra H on a unital algebra C is a map

$$H \otimes C \rightarrow C, \quad h \otimes c \mapsto h \triangleright c,$$

satisfying the following conditions:

- (H1) $1_H \triangleright c = c$ for all $c \in C$;
- (H2) $h \triangleright (cd) = (h_{(1)} \triangleright c)(h_{(2)} \triangleright d)$ for all $h \in H$ and $c, d \in C$;
- (H3) $h \triangleright (k \triangleright c) = (h_{(1)} \triangleright 1_C)(h_{(2)} k \triangleright c)$ for all $h, k \in H$ and $c \in C$;

see [13], and that such a partial action is *symmetric* if additionally

- (H4) $h \triangleright (k \triangleright c) = (h_{(1)} k \triangleright c)(h_{(2)} \triangleright 1_C)$ for all $h, k \in H$ and $c \in C$;

see [7].

Recall that the C^* -algebra \hat{A} of the discrete C^* -quantum group is a c_0 -direct sum of matrix algebras \hat{A}_α indexed by $\alpha \in \text{Irr}(\mathbb{G})$. The Hopf algebra $\mathcal{O}(\mathbb{G})$ can be identified with the subspace of all functionals $\omega \in \hat{A}'$ that vanish on \hat{A}_α for all but finitely many $\alpha \in \text{Irr}(\mathbb{G})$, and then

$$\Delta(\omega)(\hat{a} \otimes \hat{b}) = \omega(\hat{a}\hat{b}) \quad \text{and} \quad (v\omega)(\hat{a}) = (v \otimes \omega)(\hat{a})$$

for all $v, \omega \in \mathcal{O}(\mathbb{G})$ and $\hat{a}, \hat{b} \in \hat{A}$.

5.1. Theorem. *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group and let δ be a counital partial coaction of the discrete dual $(\hat{A}, \hat{\Delta})$ on a unital C^* -algebra C . Then the formula*

$$v \otimes c \mapsto v \triangleright c = (\text{id}_C \otimes v)(\delta(c)) \quad (v \in \mathcal{O}(\mathbb{G}), c \in C)$$

defines a symmetric partial action of the Hopf algebra $\mathcal{O}(\mathbb{G})$ on C .

Proof. Condition (H1) holds because the unit of $\mathcal{O}(\mathbb{G})$, regarded as a functional on \hat{A} , is the counit. Let $v, \omega \in \mathcal{O}(\mathbb{G})$ and $c, d \in C$. Choose central projections $p, q \in \hat{A}$ such that $v(p\hat{a}) = v(\hat{a})$, $\omega(\hat{a}) = \omega(q\hat{a})$ and $v_{(1)}(\hat{a})v_{(2)}(\hat{b}) = v_{(1)}(p\hat{a})v_{(2)}(q\hat{b})$ for all $\hat{a}, \hat{b} \in \hat{A}$. Then

$$v \triangleright cd = (\text{id}_C \otimes v)((1_C \otimes p)\delta(c)\delta(d)(1_C \otimes p)).$$

Since $(1_C \otimes p)\delta(c)$ and $\delta(d)(1_C \otimes p)$ are contained in the tensor product of C with the finite-dimensional C^* -algebra $p\hat{A} + q\hat{A}$, this expression is equal to

$$(\text{id} \otimes v_{(1)})((1 \otimes p)\delta(c)) \cdot (\text{id} \otimes v_{(2)})(\delta(d)(1 \otimes p)) = (v_{(1)} \triangleright c)(v_{(2)} \triangleright d).$$

Thus, condition (H2) is satisfied. Likewise,

$$\begin{aligned} v \triangleright (\omega \triangleright c) &= (\text{id}_C \otimes v \otimes \omega)((\delta \otimes \text{id}_{\hat{A}})\delta(c)) \\ &= (\text{id}_C \otimes v \otimes \omega)((1_C \otimes p \otimes q)(\delta(1_C) \otimes 1_{\hat{A}})(\text{id}_C \otimes \hat{\Delta})(\delta(c))), \end{aligned}$$

and a similar argument as above shows that this expression is equal to

$$(\text{id}_C \otimes v_{(1)})(\delta(1_C))(\text{id}_C \otimes (v_{(2)} \otimes \omega) \circ \hat{\Delta})(\delta(c)) = (v_{(1)} \triangleright 1_C) \cdot (v_{(2)} \omega \triangleright c).$$

Therefore, condition (H3) holds as well, and a similar argument proves (H4). \square

Next, we consider partial coactions of the compact C^* -quantum group (A, Δ) , and relate them to partial coactions of the Hopf algebra $\mathcal{O}(\mathbb{G})$. Recall that a *partial coaction* of a Hopf algebra H on a unital algebra C is a homomorphism

$$\delta: C \mapsto C \otimes H$$

satisfying the following conditions,

(CH1) $(\delta \otimes \text{id}_H)(\delta(c)) = (\delta(1_C) \otimes 1_H) \cdot (\text{id}_C \otimes \Delta_H)(\delta(c))$ for all $c \in C$, and

(CH2) $(\text{id}_C \otimes \varepsilon_H)(\delta_0(c)) = c$ for all $c \in C$;

see [12].

5.2. Theorem. *Let δ be a partial coaction of a compact C^* -quantum group $\mathbb{G} = (A, \Delta)$ on a unital C^* -algebra C . Then the following conditions are equivalent:*

- (1) δ is weakly continuous, $\delta(1_C)$ lies in the algebraic tensor product $C \otimes \mathcal{O}(\mathbb{G})$ and $(\text{id}_C \otimes \varepsilon)(\delta(1_C)) = 1_C$, where ε denotes the counit of $\mathcal{O}(\mathbb{G})$.
- (2) δ restricts to a partial coaction of $\mathcal{O}(\mathbb{G})$ on a unital dense $*$ -subalgebra C_0 of C .

Proof. Denote by $\mathcal{O}(\hat{\mathbb{G}}) \subseteq \hat{A}$ the algebraic direct sum of the matrix algebras \hat{A}_α associated to all $\alpha \in \text{Irr}(\mathbb{G})$, and recall that we can canonically identify $\mathcal{O}(\mathbb{G})$ with a subspace of A' .

(1) \Rightarrow (2): By Remark 3.3, the subspace $C_0 = \mathcal{O}(\hat{\mathbb{G}}) \triangleright C$ of C is dense. We show that $C_0 \subseteq C$ is a subalgebra. Let $c, d \in C$ and $v, \omega \in \mathcal{O}(\mathbb{G})$. Then

$$(v \triangleright c)(\omega \triangleright d) = (\text{id}_C \otimes v \otimes \omega)(\delta(c)_{12}\delta(d)_{13}),$$

where we use the leg notation on $\delta(c)$ and $\delta(d)$. Now, we find finitely many $v'_i, \omega'_i \in \mathcal{O}(\hat{\mathbb{G}})$ such that

$$v(a)\omega(b) = \sum_i (v'_i \otimes \omega'_i)((a \otimes 1_A)\Delta(b))$$

for all $a, b \in A$, and then

$$\begin{aligned} (v \triangleright c)(\omega \triangleright d) &= \sum_i (\text{id}_C \otimes v'_i \otimes \omega'_i)((\delta(c) \otimes 1_A)(\text{id} \otimes \Delta)(\delta(d))) \\ &= \sum_i (\text{id}_C \otimes v'_i \otimes \omega'_i)(\delta \otimes \text{id}_A)((c \otimes 1_A)\delta(d)) = \sum_i v'_i \triangleright (c(\omega'_i \triangleright d)) \in C_0. \end{aligned}$$

Next, we show that $\delta(C_0)$ is contained in the algebraic tensor product $C \odot \mathcal{O}(\mathbb{G})$. Let $\omega \in \mathcal{O}(\hat{\mathbb{G}})$ and $c \in C$. Since $\mathcal{O}(\mathbb{G})$ has a basis of elements $(u_{i,j}^\alpha)_{\alpha,i,j}$ satisfying $\Delta(u_{i,j}^\alpha) = \sum_k u_{ik}^\alpha \otimes u_{kj}^\alpha$ [31, Proposition 5.1], we can find finitely many $v_1, \dots, v_n \in \mathcal{O}(\hat{\mathbb{G}})$ and $a_1, \dots, a_n \in \mathcal{O}(\mathbb{G})$ such that

$$(\text{id}_A \otimes \omega)(\Delta(b)) = \sum_{i=1}^n v_i(b)a_i$$

for all $b \in \mathcal{O}(\mathbb{G})$, and then

$$\begin{aligned} \delta(\omega \triangleright c) &= (\text{id}_C \otimes \text{id}_A \otimes \omega)(\delta \otimes \text{id}_A)\delta(c) \\ &= \delta(1_C)(\text{id}_C \otimes (\text{id}_A \otimes \omega)\Delta)\delta(c) = \delta(1_C) \cdot \sum_{i=1}^n (v_i \triangleright c) \otimes a_i \end{aligned}$$

lies in the algebraic tensor product of C with $\mathcal{O}(\mathbb{G})$. Using a basis for $\mathcal{O}(\hat{\mathbb{G}})$ consisting of functionals $(\phi_{i,j}^\alpha)_{\alpha,i,j}$ such that $\phi_{i,j}^\alpha(u_{k,l}^\beta) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}$, see [31, §6], we see that $\delta(C_0)$ is contained in the algebraic tensor product $C_0 \otimes \mathcal{O}(\mathbb{G})$.

To finish the proof, note that with ω, c as above, (3.4) implies

$$\varepsilon \triangleright (\omega \triangleright c) = (\text{id}_C \otimes \varepsilon)(\delta(1_C)) \cdot (\omega \triangleright c) = \omega \triangleright c.$$

(2) \Rightarrow (1): Since $C_0 \subseteq C$ is dense, the unit of C_0 has to be 1_C , whence $\delta(1_C)$ lies in the algebraic tensor product $C \otimes \mathcal{O}(\hat{\mathbb{G}})$ and $(\text{id} \otimes \varepsilon)\delta(1_C) = 1_C$. To prove weak continuity, we show that for every $c \in C_0$, there exists some $\omega \in A'$ such that $\omega \triangleright c = c$. So, take $c \in C_0$ and write $\delta(c) = \sum_{i=1}^n d_i \otimes a_i$ with $d_i \in C_0$ and $a_i \in \mathcal{O}(\mathbb{G})$. By Hahn-Banach, the restriction of ε to the finite-dimensional subspace of A spanned by a_1, \dots, a_n extends to a bounded linear functional $\omega \in A'$ that satisfies $\omega \triangleright c = \varepsilon \triangleright c = c$. \square

6. RESTRICTION

Like partial actions of groups and partial (co)actions of Hopf algebras, partial coactions of C^* -bialgebras can be obtained from non-partial ones by restriction.

6.1. Definition. Let δ_B be a partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra B . We call a C^* -subalgebra $C \subseteq B$ weakly invariant if

$$\delta_B(C)(C \otimes A) \subseteq C \otimes A,$$

and strongly invariant if the embedding $C \hookrightarrow B$ is strict and $\delta_B(C) \subseteq M(C \otimes A) \subseteq M(B \otimes A)$.

Note here that if the embedding $C \hookrightarrow B$ is strict, then the embedding $C \otimes A \hookrightarrow B \otimes A$ is strict as well and extends to an embedding $M(C \otimes A) \hookrightarrow M(B \otimes A)$.

- 6.2. Remark.** (1) Every ideal $C \subseteq B$ is weakly invariant, but not necessarily strongly invariant.
 (2) A corner $C \subseteq B$ is strongly invariant if and only if $1_C \in M(C) \subseteq M(B)$ is strongly invariant in the sense that

$$\delta_B(1_C) = \delta_B(1_C)(1_C \otimes 1_A),$$

as one can easily check. If one thinks of elements of $M(B)$ and $M(B \otimes A)$ as 2×2 -matrices with respect to the Pierce decomposition $B = 1_C B + (1_B - 1_C)B$, then strong invariance of C means that $\delta_B(C)$ is contained in the upper left corner, while weak invariance of C means that the off-diagonal part of $\delta_B(C)$ vanishes.

6.3. Example. Suppose that δ_B is the partial coaction corresponding to a disconnected partial action $((p_g)_{g \in \Gamma}, (\theta_g)_{g \in \Gamma})$ of a discrete group Γ on a C^* -algebra B as in Proposition 4.2, and that $C \subseteq B$ is a direct summand. Then C is automatically weakly invariant, but strongly invariant if and only if $\theta_g(p_{g^{-1}}C) \subseteq C$ for all $g \in \Gamma$.

Evidently, partial coactions can be restricted to strongly invariant C^* -subalgebras. Restriction to weakly invariant C^* -subalgebras is a bit more delicate unless the embedding of the C^* -subalgebra is strict.

6.4. Proposition. Let δ_B be a partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra B and let $C \subseteq B$ be a weakly invariant C^* -subalgebra. Then:

- (1) δ_B restricts to a $*$ -homomorphism $\delta_C: C \rightarrow M(C \otimes A)$.
 (2) If the embedding $C \hookrightarrow B$ is strict, then the composition of δ_C with the embedding of $M(C \otimes A)$ into $M(B \otimes A)$ is strict and

$$\delta_C(c) = \delta_B(c)(1_C \otimes 1_A) \quad (c \in C).$$

- (3) If δ_C is strict, then it is a partial coaction of (A, Δ) on C .

Proof. (1) This follows immediately from the definition.

(2) Suppose that the embedding $C \hookrightarrow B$ is strict. Then so is its composition with δ_B and hence also δ_C . To prove the formula given for $\delta_C(c)$, choose a bounded approximate unit $(u_\nu)_\nu$ for C , and note that $\delta_C(c)(u_\nu \otimes 1_A) = \delta_B(c)(u_\nu \otimes 1_A)$ converges strictly to $\delta_C(c)$ in $M(C \otimes A)$ and to $\delta_B(c)(1_C \otimes 1_A)$ in $M(B \otimes A)$.

- (3) Let $(u_\nu)_\nu$ be as above and let $c, c' \in C$. Then by definition of δ_C ,

$$\begin{aligned} & (c' \otimes 1_A \otimes 1_A) \cdot (\delta_C \otimes \text{id}_A)(\delta_C(c)(u_\nu \otimes 1_A)) \\ &= (c' \otimes 1_A \otimes 1_A) \cdot (\delta_C \otimes \text{id}_A)(\delta_B(c)(u_\nu \otimes 1_A)) \\ &= (c' \otimes 1_A \otimes 1_A) \cdot (\delta_B \otimes \text{id}_A)(\delta_B(c)) \cdot (\delta_C(u_\nu) \otimes 1_A) \\ &= (\text{id}_C \otimes \Delta)((c' \otimes 1_A)\delta_B(c)) \cdot (\delta_C(u_\nu) \otimes 1_A) \\ &= (c' \otimes 1_A \otimes 1_A) \cdot (\text{id}_C \otimes \Delta)(\delta_C(c)) \cdot (\delta_C(u_\nu) \otimes 1_A). \end{aligned}$$

Since $c' \in C$ was arbitrary, we can conclude that

$$(\delta_C \otimes \text{id}_A)(\delta_C(c)(u_\nu \otimes 1_A)) = (\text{id}_C \otimes \Delta)(\delta_C(c)) \cdot (\delta_C(u_\nu) \otimes 1_A).$$

As ν tends to infinity, $\delta_C(c)(u_\nu \otimes 1_A)$ converges strictly to $\delta_C(c)$, and since δ_C and hence also $\delta_C \otimes \text{id}_A$ are strict, the left hand side converges to $(\delta_C \otimes \text{id}_A)\delta_C(c)$ and the right hand side converges to $(\text{id}_C \otimes \Delta)(\delta_C(c))(\delta_C(1_C) \otimes 1_A)$. \square

6.5. Remark. (1) As a corollary, a (partial) coaction on a C^* -algebra C restricts to a partial coaction on every direct summand of C .

(2) The restriction δ_C can be strict without the embedding $C \hookrightarrow B$ being strict, for example, this is the case if δ_B is the trivial coaction $b \mapsto b \otimes 1_A$ and $C \subseteq B$ is a closed ideal that is not a direct summand.

6.6. Example. Let $\mathbb{G} = (A, \Delta)$ be a discrete quantum group, so that A is a c_0 -sum of matrix algebras A_α with $\alpha \in \text{Irr}(\hat{\mathbb{G}})$. Then for every subset $\mathcal{J} \subseteq \text{Irr}(\hat{\mathbb{G}})$, the restriction of Δ to the c_0 -sum $A_{\mathcal{J}} := \bigoplus_{\alpha \in \mathcal{J}} A_\alpha$ yields a partial coaction. But if \mathcal{J} is non-trivial, then $A_{\mathcal{J}}$ is not strongly invariant: if $\alpha \notin \mathcal{J}$ and $\gamma \in \mathcal{J}$, then $\alpha \otimes (\alpha^\dagger \otimes \gamma)$, where α^\dagger denotes the dual of α , contains γ , and hence $\Delta(A_\gamma)(A_\alpha \otimes 1) \neq 0$.

Closely related to the concept of restriction is the notion of a morphism of partial coactions.

6.7. Definition. Let δ_B and δ_C be partial coactions of a C^* -bialgebra (A, Δ) on C^* -algebras B and C , respectively. A strong morphism from δ_C to δ_B is a strict $*$ -homomorphism $\pi: C \rightarrow M(B)$ satisfying

$$(\pi \otimes \text{id}_A)\delta_C(c) = \delta_B(\pi(c)) \quad (c \in C).$$

A weak morphism from δ_C to δ_B is a $*$ -homomorphism $\pi: C \rightarrow M(B)$ satisfying

$$(\pi \otimes \text{id}_A)(\delta_C(c)(c' \otimes a)) = \delta_B(\pi(c))(\pi(c') \otimes a) \quad (c, c' \in C, a \in A).$$

We call such a weak or strong morphism π proper if $\pi(C) \subseteq B$.

Evidently, partial coactions with strong morphisms or with proper weak morphisms as above form categories.

6.8. Remark. (1) Clearly, π is a strong or a weak morphism if and only if

$$(6.1) \quad \pi(\omega \triangleright c) = \omega \triangleright \pi(c) \quad \text{or} \quad \pi(\omega \triangleright c)\pi(c') = (\omega \triangleright \pi(c))\pi(c')$$

respectively, for all $\omega \in A'$ and $c, c' \in C$.

(2) If π is a weak or a strong morphism and proper, then its image is weakly or strongly invariant, respectively.

(3) Suppose that δ_B is a partial coaction of (A, Δ) on a C^* -algebra B and that $C \subseteq B$ is a C^* -subalgebra that is weakly or strongly invariant. If the embedding $C \hookrightarrow B$ is strict, then this embedding is a weak or a strong morphism with respect to the restriction of δ_B to C defined above.

Let us look at the special case of partial coactions associated to disconnected partial group actions.

6.9. Proposition. *Let B and C be two C^* -algebras with disconnected partial actions $((p_g)_g, (\beta_g)_g)$ and $((q_g)_g, (\gamma_g)_g)$, respectively, of a discrete group Γ . With respect to the associated partial coactions of $C_0(\Gamma)$, a strict $*$ -homomorphism $\pi: B \rightarrow M(C)$ is a strong morphism if and only if*

$$(6.2) \quad \pi(p_g) = q_g \pi(1_C) \quad \text{and} \quad \pi \circ \beta_g \subseteq \gamma_g \circ \pi \quad \text{for all } g \in \Gamma,$$

and a weak morphism if and only if

$$(6.3) \quad \pi(1_C) \gamma_g(q_{g^{-1}} \pi(1_C)) = \pi(p_g) = \gamma_g(\pi(p_{g^{-1}})) \quad \text{and} \quad \pi \circ \beta_g \subseteq \gamma_g \circ \pi \quad \text{for all } g \in \Gamma.$$

Proof. Denote the partial coactions by δ_B and δ_C .

(1) Suppose that π is a strong morphism. Then the definition of δ_B and δ_C implies

$$(6.4) \quad (\pi \circ \beta_g)(p_{g^{-1}} b) = (\pi \otimes \text{ev}_g) \delta_B(b) = (\text{id}_C \otimes \text{ev}_g) \delta_C(\pi(b)) = \gamma_g(q_{g^{-1}} \pi(b))$$

for all $g \in \Gamma$ and $b \in B$. Taking $b = 1_C$ or $b = p_{g^{-1}}$, we conclude that

$$\gamma_g(q_{g^{-1}} \pi(p_{g^{-1}})) = \pi(p_g) = \gamma_g(q_{g^{-1}} \pi(1_C)),$$

in particular, $\pi(p_g) q_g = \pi(p_g)$. We use this relation on the left hand side above, apply $\gamma_{g^{-1}}$, and get $\pi(p_g) = q_g \pi(1_C)$. Moreover, $\pi(p_g B) \subseteq q_g C$, and (6.4) implies $\pi \circ \beta_g \subseteq \gamma_g \circ \pi$.

Conversely, the first relation in (6.2) implies $q_{g^{-1}} \pi(1_C - p_{g^{-1}}) = 0$, whence both sides in (6.4) are zero for all $b \in (1_C - p_{g^{-1}}) B$, and the second relation in (6.2) implies that (6.4) holds for all $b \in p_{g^{-1}} B$. Combined, (6.2) implies $(\pi \otimes \text{id}) \delta_B = \delta_C \circ \pi$.

(2) Suppose that π is a strict weak morphism. As in (1), we find that

$$(6.5) \quad (\pi \circ \beta_g)(p_{g^{-1}} b) = \pi(1_C) \gamma_g(q_{g^{-1}} \pi(b))$$

for all $g \in \Gamma$ and $b \in B$, and similar arguments as in (1) yield the first equation in (6.3). Now, we apply $\gamma_{g^{-1}}$ to this relation and find that

$$\gamma_{g^{-1}}(\pi(p_g)) = \gamma_{g^{-1}}(q_g \pi(1_C)) \pi(1_C) = \pi(p_{g^{-1}}).$$

In particular, this relation and (6.5) imply the second relation in (6.3).

Conversely, (6.3) implies that both sides of (6.5) coincide for all $b \in p_{g^{-1}} B$, and that for all $b \in (1_C - p_{g^{-1}}) B$,

$$\pi(1_C) \gamma_g(q_{g^{-1}} \pi(1_C - p_{g^{-1}})) = \pi(p_g) - \pi(p_g) = 0,$$

whence both sides of (6.5) are zero for all $b \in (1_C - p_{g^{-1}}) B$. But this implies that $(\pi \otimes \text{id}) \circ \delta_B = (\pi(1_C) \otimes 1_A)(\delta_C \circ \pi)$. \square

7. DILATIONS

Let (A, Δ) be a C^* -bialgebra. Given a partial coaction of (A, Δ) , a natural and important question is whether it can be identified with as the restriction of a coaction to a weakly invariant C^* -subalgebra as in Proposition 6.4.

7.1. Definition. *Let δ_C be a partial coaction of (A, Δ) on a C^* -algebra C . A dilation of δ_C consists of a C^* -algebra B , a coaction δ_B of (A, Δ) on B , and an embedding $\iota: C \hookrightarrow B$ that is a weak morphism from δ_C to δ_B , that is, satisfies*

$$\delta_B(\iota(c))(\iota(c') \otimes a) = (\iota \otimes \text{id}_A)(\delta_C(c)(c' \otimes a)) \quad (c, c' \in C, a \in A).$$

7.2. Example (Disconnected partial actions of groups). Let C be a C^* -algebra with a disconnected partial action $((p_g)_g, (\theta_g)_g)$ of a discrete group Γ , and consider the associated partial coaction δ_C of $C_0(\Gamma)$ as in Proposition 4.2.

A dilation of δ_C is given by a C^* -algebra B with a coaction of $C_0(\Gamma)$, that is, by an action $(\alpha_g)_{g \in \Gamma}$ of Γ on B , and an embedding $C \hookrightarrow B$ that is a weak morphism. Suppose that this embedding is strict. By Proposition 6.9, it is a weak morphism if and only if

$$p_g = 1_C \alpha_g(1_C) \quad \text{and} \quad \theta_g = \alpha_g|_{p_g C} \quad (g \in \Gamma).$$

In particular, 1_C commutes with $\alpha_g(1_C)$ for each $g \in \Gamma$. We claim that our partial action coincides with the set-theoretic restriction $((D_g)_g, (\alpha_g|_{D_g})_g)$ of α to C , where $D_g = \alpha_g(C) \cap C$ for each $g \in \Gamma$. Indeed, for every element $c \in D_g$ with $0 \leq c \leq 1_C$, we have $c \leq 1_C$ and $\alpha_g^{-1}(c) \leq 1_C$, whence $c \leq \alpha_g(1_C)1_C = p_g$ and $c \in p_g C$. On the other hand, if $c \in p_g C$, then $\alpha_{g^{-1}}(c) = \theta_{g^{-1}}(c) \in C$ and hence $c \in \alpha_g(C) \cap C = D_g$.

Conversely, suppose that α is an action of Γ on a C^* -algebra B that contains C and that α is a dilation in the usual sense, so that $C \subseteq B$ is an ideal, $p_g C = \alpha_g(C) \cap C$ and $\theta_g = \alpha_g|_{p_g C}$ for each $g \in \Gamma$. If the embedding $C \subseteq B$ is strict, then C is a direct summand, that is, $C = 1_C B$, and then $\alpha_g(C) \cap C = \alpha_g(1_C)1_C$ for each $g \in \Gamma$, so that the coaction δ_B corresponding to α is a dilation of δ_C .

The main question is, of course, which partial coactions do have a dilation. We start with a necessary condition.

7.3. Definition. We call a partial coaction δ_C of (A, Δ) on a C^* -algebra C regular if

$$(7.1) \quad (\text{id}_C \otimes \Delta)(\delta_C(C)) \cdot (1_C \otimes 1_A \otimes A) \subseteq M(C \otimes A) \otimes A.$$

7.4. Example. (1) Every coaction is easily seen to be regular.

(2) The question of regularity arises only if C is non-unital, because every partial coaction on a unital C^* -algebra is regular.

(3) If A is a direct sum of matrix algebras, for example, if (A, Δ) is a discrete quantum group, then every partial coaction of (A, Δ) is regular.

Regularity is necessary for the existence of a dilation with a strict embedding:

7.5. Lemma. If a partial coaction has a dilation (B, δ_B, ι) , where ι is strict, then the partial coaction is regular.

Proof. Suppose that δ_C is a partial coaction of (A, Δ) on a C^* -algebra C with a dilation (B, δ_B, ι) . It suffices to show that the product

$$(\iota \otimes \text{id}_A \otimes \text{id}_A)((\text{id}_C \otimes \Delta)\delta_C(C)) \cdot (1_B \otimes 1_A \otimes A)$$

lies in $M(B \otimes A) \otimes A$. Since ι is a weak morphism, this product is equal to

$$(\text{id}_C \otimes \Delta)(\delta_B(\iota(C))) \cdot (\iota(1_C) \otimes 1_A \otimes A),$$

which by coassociativity of δ_B can be rewritten as

$$(\delta_B \otimes \text{id}_A)(\delta_B(\iota(C))(1_B \otimes A)) \cdot (\iota(1_C) \otimes 1_A \otimes 1_A),$$

and this product lies in $M(B \otimes A) \otimes A$ because $\delta_B(\iota(C))(1_B \otimes A) \subseteq B \otimes A$. \square

If (A, Δ) is a weakly regular C^* -quantum group, for example, a compact one, and if δ_C is weakly continuous, then regularity of δ_C can be tested on the unit:

7.6. Lemma. *Let (A, Δ) be a weakly regular C^* -quantum group and let δ_C be a weakly continuous partial coaction of (A, Δ) on a C^* -algebra C such that*

$$(\text{id}_C \otimes \Delta)(\delta_C(1_C)) \cdot (1_C \otimes 1_A \otimes A) \subseteq M(C \otimes A) \otimes A.$$

Then δ_C is regular.

Proof. We use the same notation and a similar argument as in the proof of Proposition 3.5. By (3.3),

$$(\text{id}_C \otimes \Delta)(\delta_C(\omega \triangleright c)) = (\text{id}_C \otimes \Delta)(\delta_C(1_C)) \cdot (\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta^{(2)})\delta_C(c)$$

for all $\omega \in A'$ and $c \in C$, where $\Delta^{(2)} = (\Delta \otimes \text{id}_A)\Delta = (\text{id}_A \otimes \Delta)\Delta$. Since δ_C is weakly continuous, we can conclude that $[(\text{id}_C \otimes \Delta)\delta_C(C) \cdot (1_C \otimes 1_A \otimes A)]$ is equal to the product of $(\text{id}_C \otimes \Delta)(\delta_C(1_C))$ with

$$[(\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)((\text{id}_C \otimes \Delta^{(2)})(\delta_C(C))(1_C \otimes 1_A \otimes A \otimes 1_A)) : \omega \in A'].$$

Similarly as in the proof of Proposition 3.5, we rewrite this space in the form

$$\begin{aligned} & [(\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)(V_{34}(\text{id}_C \otimes (\text{id}_A \otimes \pi)\Delta)(\delta_C(C))_{124}V_{34}^*(A \otimes \hat{\pi}(\hat{A}))_{34}) : \omega \in \mathcal{B}(K)_*] \\ &= [(\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes \omega)((A \otimes \hat{\pi}(\hat{A}))_{34}(\text{id}_C \otimes (\text{id}_A \otimes \pi)\Delta)(\delta_C(C))_{124}) : \omega \in \mathcal{B}(K)_*] \\ &= [((\text{id}_C \otimes \text{id}_A \otimes \omega \circ \pi)(\text{id}_C \otimes \Delta)\delta_C(C)) \otimes A : \omega \in \mathcal{B}(K)_*] \\ &\subseteq M(C \otimes A) \otimes A. \end{aligned}$$

Summarising, we find that

$$(\text{id}_C \otimes \Delta)\delta_C(C) \cdot (1_C \otimes 1_A \otimes A) \subseteq (\text{id}_C \otimes \Delta)\delta_C(1_C) \cdot (M(C \otimes A) \otimes A).$$

By assumption on $\delta_C(1_C)$, the right hand side lies in $M(C \otimes A) \otimes A$. \square

For partial actions of a group G on a set or space X , a canonical dilation can be constructed as a certain quotient of the product $X \times G$; see [2] or [17, Theorem 3.5, Proposition 5.5], and Example 8.7. We now give a dual construction. Although this one will be improved upon in the next section, we decided to include it for instructive purpose, see also Example 8.7.

From now on, we will almost always assume the C^* -algebra underlying our C^* -bialgebra to have the slice map property, which holds, for example, if it is nuclear; see 2.1.

7.7. Proposition. *Let δ_C be an injective, regular partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra C and suppose that A has the slice map property. Denote by $C \boxtimes A \subseteq M(C \otimes A)$ the subset of all x satisfying the following conditions:*

- (1) $[x, \delta_C(1_C)] = 0$;
- (2) $(\delta_C \otimes \text{id}_A)(x) = (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)(x) = (\text{id}_C \otimes \Delta)(x)(\delta_C(1_C) \otimes 1_A)$;
- (3) $x(1_C \otimes A)$ and $(1_C \otimes A)x$ lie in $C \otimes A$;
- (4) $(\text{id}_C \otimes \Delta)(x)(1_C \otimes 1_A \otimes A)$ and $(1_C \otimes 1_A \otimes A)(\text{id}_C \otimes \Delta)(x)$ lie in $M(C \otimes A) \otimes A$.

Then $C \boxtimes A$ is a C^ -algebra, $\text{id}_C \otimes \Delta$ restricts to a coaction of (A, Δ) on $C \boxtimes A$, and $(C \boxtimes A, \text{id}_C \otimes \Delta, \delta_C)$ is a dilation of δ_C .*

Proof. Clearly, $C \boxtimes A$ is a C^* -algebra. It contains $\delta_C(C)$ by (3.1) and regularity of δ_C . Next, we need to show that

$$(\text{id}_C \otimes \Delta)(C \boxtimes A)(1_C \otimes 1_A \otimes A) \subseteq (C \boxtimes A) \otimes A.$$

Condition (4) implies that the left hand side is contained in $M(C \otimes A) \otimes A$. Since A has the slice map property, it suffices to show that for every $y \in C \boxtimes A$ and $\omega \in A'$, the element

$$x := (\text{id}_C \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta)(y) = (\text{id}_C \otimes (\text{id}_A \otimes \omega)\Delta)(y)$$

lies in $C \boxtimes A$, that is, satisfies conditions (1)–(4) above. In case of (2)–(4), we only prove the first halves of the statements, the others follow similarly.

(1) The element x commutes with $\delta_C(1_C)$ because $(\text{id}_C \otimes \Delta)(y)$ commutes with $(\delta_C(1_C) \otimes 1_A)$ by (2), applied to y .

(2) We use (1) for y and coassociativity of Δ to see that

$$\begin{aligned} (\delta_C \otimes \text{id}_A)(x) &= (\text{id}_C \otimes \text{id}_A \otimes (\text{id}_A \otimes \omega)\Delta)(\delta_C \otimes \text{id}_A)(y) \\ &= (\text{id}_C \otimes \text{id}_A \otimes (\text{id}_A \otimes \omega)\Delta)((\delta_C(1) \otimes 1_A)(\text{id}_C \otimes \Delta)(y)) \\ &= (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes (\text{id}_A \otimes \text{id}_A \otimes \omega)\Delta^{(2)})(y) \\ &= (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)(\text{id}_A \otimes (\text{id}_A \otimes \omega)\Delta)(y) \\ &= (\delta_C(1_C) \otimes 1_A)(\text{id}_C \otimes \Delta)(x). \end{aligned}$$

(3) Write $\omega = av$ with $a \in A$ and $v \in A'$ using Cohen's factorisation theorem, and let $a' \in A$. Then

$$x(1_C \otimes a') = (\text{id}_C \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta)(y)(1_C \otimes a' \otimes a)).$$

We use the relation $A \otimes A = [\Delta(A)(A \otimes A)]$ and condition (3) on y and find that $x(1_C \otimes a')$ lies in $C \otimes A$ as desired.

(4) With a, a', v as above,

$$(\text{id}_C \otimes \Delta)(x) \cdot (1_C \otimes 1_A \otimes a') = (\text{id}_C \otimes \text{id}_A \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta^{(2)})(y) \cdot (1_C \otimes 1_A \otimes a' \otimes a)).$$

We use the relation $A \otimes A = [\Delta(A)(A \otimes A)]$ again and find that

$$\begin{aligned} (\text{id}_C \otimes \Delta^{(2)})(y) \cdot (1_C \otimes 1_A \otimes a' \otimes a) \\ \in (\text{id}_C \otimes \text{id}_A \otimes \Delta)((\text{id}_C \otimes \Delta)(y) \cdot (1_C \otimes 1_A \otimes A)) \cdot (1_C \otimes 1_A \otimes A \otimes A). \end{aligned}$$

Condition (4), applied to y , implies that the expression above lies in $M(C \otimes A) \otimes A \otimes A$. Slicing the last factor with v , we get $(\text{id}_C \otimes \Delta)(x) \cdot (1_C \otimes 1_A \otimes a') \in M(C \otimes A) \otimes A$. \square

7.8. Example (Case of a partial group action). Consider the partial coaction δ_C associated to a disconnected partial action $((p_g), (\theta_g)_g)$ of a discrete group Γ on a C^* -algebra C . Identify $M(C \otimes C_0(\Gamma))$ with $C_b(\Gamma; M(C))$ and let $f \in C_b(\Gamma; M(C))$. Then conditions (1) and (4) in Proposition 7.7 are automatically satisfied by f , condition (3) is equivalent to $f \in C_b(\Gamma; C)$, and condition (2) corresponds to the invariance condition

$$\theta_g(p_{g^{-1}}f(h)) = p_gf(gh) \quad (g, h \in \Gamma).$$

In particular, if $C = C_0(X)$ for some locally compact, Hausdorff space X , then each p_g is the characteristic function of some clopen $D_g \subseteq X$, each θ_g is the pull-back along some homeomorphism $\alpha_{g^{-1}}: D_g \rightarrow D_{g^{-1}}$, and the invariance condition above translates into

$$f(x, gh) = f(\alpha_{g^{-1}}(x), h) \quad (g, h \in \Gamma, x \in D_g),$$

so that f descends to the quotient space of $X \times \Gamma$ with respect to the equivalence relation given by $(x, gh) \sim (\alpha_{g^{-1}}(x), h)$ for all $g, h \in \Gamma$ and $x \in D_g$. This space is, up to the reparameterization $(x, g) \mapsto (g^{-1}, x)$, the globalization of the partial action $((D_g)_g, (\alpha_g)_g)$ of Γ on X , see [17, Theorem 3.5, Proposition 5.5], and $C_0(X) \boxtimes C_0(\Gamma)$ can be identified with a C^* -subalgebra of $C_b((X \times \Gamma)/\sim)$.

8. MINIMAL DILATIONS

Among all dilations of a fixed partial coaction δ_C of a C^* -bialgebra (A, Δ) , we now single out a universal one, which we call the globalization of δ_C . More precisely, we show that (1) every dilation of δ_C contains one that is minimal in a natural sense, and (2) that all such minimal dilations are isomorphic. We need to assume, however, that δ_C is regular and injective, that A has the slice map property, and, for (2), that (A, Δ) is a C^* -quantum group. Finally, we give a categorical interpretation of these results.

8.1. Definition. Let δ_C be a partial coaction of (A, Δ) on a C^* -algebra C . We call a dilation (B, δ_B, ι) of δ_C minimal if $\iota(C)$ and $A' \triangleright \iota(C)$ generate B as a C^* -algebra.

8.2. Remark. Let (B, δ_B, ι) be a minimal dilation of a partial coaction δ_C of (A, Δ) on some C^* -algebra C . Then $\iota(C) \subseteq B$ is an ideal because $\iota(C)(A' \triangleright \iota(C)) = \iota(C)\iota(A' \triangleright C) \subseteq \iota(C)$ by (6.1). If, moreover, ι is strict, then $\iota(C)$ is a direct summand of B .

8.3. Example. If, in the situation above, (A, Δ) is the C^* -bialgebra of functions on a discrete group Γ , then the coaction δ_B corresponds to an action α of Γ on B , and the dilation is minimal if and only if $\sum_{g \in \Gamma} \alpha_g(\iota(C))$ generates B as a C^* -algebra.

Every dilation contains a minimal one:

8.4. Proposition. Let δ_C be a partial coaction of (A, Δ) on a C^* -algebra C with a dilation (B, δ_B, ι) , and suppose that A has the slice map property. Denote by $B_0 \subseteq B$ the C^* -subalgebra generated by $\iota(C)$ and $A' \triangleright \iota(C)$.

- (1) δ_B restricts to a coaction δ_{B_0} on B_0 , and $(B_0, \delta_{B_0}, \iota)$ is a minimal dilation of δ_C .
- (2) If (A, Δ) is a weakly regular C^* -quantum group and δ_C is weakly continuous, then $[A' \triangleright \iota(C)] \subseteq B$ is a C^* -algebra. If additionally ι is strict, then $B_0 = [(\iota(C) + \mathbb{C}1_B)(A' \triangleright \iota(C))]$.

Proof. (1) To prove the first assertion, we only need to show that

$$\delta_B(\iota(C))(1_B \otimes A) \subseteq B_0 \otimes A \quad \text{and} \quad \delta_B(A' \triangleright \iota(C))(1_B \otimes A) \subseteq B_0 \otimes A.$$

But for all $c \in C$, $v, \omega \in A'$, both $(\text{id} \otimes \omega)(\delta_B(\iota(c))) = \omega \triangleright \iota(c)$ and $(\text{id} \otimes \omega)(\delta_B(v \triangleright \iota(c))) = \omega v \triangleright \iota(c)$ lie in B_0 . Since A has the slice map property, the desired inclusions follow.

(2) We follow the proof of [9, Proposition 5.7], using the same notation and manipulations as in the proof of Proposition 3.5. To shorten the notation, let $U := (\pi \otimes \text{id}_{\hat{\pi}(\hat{A})})(V)$

and $\delta_\pi := (\text{id}_B \otimes \pi) \circ \delta_B \circ \iota$. Then by (3.3),

$$\begin{aligned}
[A' \triangleright \iota(C)] &= [A' \triangleright \iota(C(A' \triangleright C))] \\
&= [(\text{id}_B \otimes v \otimes \omega)((\delta_B \otimes \text{id}_A)((C \otimes 1_A)\delta_C(C))) : v, \omega \in A'] \\
&= [(\text{id}_B \otimes v \otimes \omega)((\delta_B \otimes \text{id}_A)((C \otimes 1_A)\delta_B(C))) : v, \omega \in A'] \\
&= [(\text{id}_B \otimes v \circ \pi \otimes \omega \circ \pi)((\delta_B(C) \otimes 1_A)(\text{id}_B \otimes \Delta)\delta_B(C)) : v, \omega \in \mathcal{B}(K)_*] \\
&= [(\text{id}_B \otimes v \otimes \omega)(\delta_\pi(C)_{12}U_{23}\delta_\pi(C)_{13}U_{23}^*) : v, \omega \in \mathcal{B}(K)_*] \\
&= [(\text{id}_B \otimes v \otimes \omega)(\delta_\pi(C)_{12}U_{23}\delta_\pi(C)_{13}(\pi(A) \otimes \hat{\pi}(\hat{A}))_{23}) : v, \omega \in \mathcal{B}(K)_*] \\
&= [(\text{id}_B \otimes v \otimes \omega)(\delta_\pi(C)_{12}(\pi(A) \otimes \hat{\pi}(\hat{A}))_{23}\delta_\pi(C)_{13}) : v, \omega \in \mathcal{B}(K)_*] \\
&= [(A' \triangleright \iota(C))(A' \triangleright \iota(C))].
\end{aligned}$$

Thus, $[A' \triangleright \iota(C)]$ is a C^* -algebra. If ι is strict so that $\iota(1_C)$ is well-defined, then this C^* -algebra commutes with $\iota(1_C)$, and by (6.1) the product is $[\iota(A' \triangleright C)] = \iota(C)$. This proves the last assertion concerning B_0 . \square

If we apply Proposition 8.4 to the canonical dilation $(C \boxtimes A, \text{id}_C \otimes \Delta, \delta_C)$ constructed in Proposition 7.7, we obtain the following dilation:

8.5. Theorem. *Let (A, Δ) be a C^* -bialgebra, where A has the slice map property, and let δ_C be an injective, regular partial coaction of (A, Δ) on a C^* -algebra C . Denote by $\mathfrak{G}(C) \subseteq M(C \otimes A)$ the C^* -subalgebra generated by*

$$\{(\text{id}_C \otimes \text{id}_C \otimes \omega)(\text{id}_C \otimes \Delta)\delta_C(c) : \omega \in A', c \in C\} \quad \text{and} \quad \delta_C(C).$$

Then $\text{id}_C \otimes \Delta$ restricts to a partial coaction on $\mathfrak{G}(C)$ and

$$\mathfrak{G}(\delta_C) := (\mathfrak{G}(C), \text{id}_C \otimes \Delta, \delta_C)$$

is a minimal dilation of δ_C .

Proof. By a similar argument as in the proof of Proposition 7.7, we only need to show that for every $c \in C$ and $v, \omega \in A'$, the elements

$$(\text{id}_C \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta)\delta_C(c))$$

and

$$(\text{id}_C \otimes \text{id}_A \otimes v)((\text{id}_C \otimes \Delta)((\text{id}_C \otimes \text{id}_A \otimes \omega)(\text{id}_C \otimes \Delta)\delta_C(c)))$$

lie in $\mathfrak{G}(C)$. In the first case, this is trivially true, and in the second case, one finds that the element is equal to $d = (\text{id}_C \otimes \text{id}_A \otimes v\omega)((\text{id}_C \otimes \Delta)\delta_C(C)) \in \mathfrak{G}(C)$. \square

8.6. Remark. Suppose that (A, Δ) and δ_C are as above.

- (1) Beware that δ_C is strict as a map from C to $M(C \otimes A)$, but this does not imply that δ_C is strict as a map from C to $\mathfrak{G}(C)$.
- (2) If δ_C is weakly continuous, then (3.3) implies that $\mathfrak{G}(C) \subseteq M(C \otimes A)$ is equal to the C^* -subalgebra generated by $\{(\text{id}_C \otimes \text{id}_C \otimes \omega)(\text{id}_C \otimes \Delta)\delta_C(c) : \omega \in A', c \in C\}$ and $\delta_C(1_C)$.

8.7. Example (Case of a partial group action). Consider the partial coaction δ_C associated to a disconnected partial action $((p_g), (\theta_g)_g)$ of a discrete group Γ on a C^* -algebra C , and identify $M(C \otimes C_0(\Gamma))$ with $C_b(\Gamma; M(C))$. In that case, $\mathfrak{G}(C)$ is the C^* -algebra generated by all functions of the form

$$f_{c,h} = (\text{id}_C \otimes \text{id}_{C_0(\Gamma)} \otimes \text{ev}_h)(\text{id}_C \otimes \Delta)\delta_C(c): g \mapsto \theta_{gh}(p_{h^{-1}g^{-1}}c),$$

where $c \in C$ and $g, h \in \Gamma$. The action ρ of Γ corresponding to the coaction $\text{id}_C \otimes \Delta$ is given by right translation of functions, whence $\rho_{h'}(f_{c,h}) = f_{c,h'h}$ for all $h' \in \Gamma$.

We shall use the following notion of a morphism between dilations:

8.8. Definition. Let δ_C be a partial coaction of a C^* -bialgebra (A, Δ) on some C^* -algebra C . A morphism between dilations $\mathcal{B} = (B, \delta_B, \iota^B)$ and $\mathcal{D} = (D, \delta_D, \iota^D)$ of δ_C is a $*$ -homomorphism $\phi: B \rightarrow D$ satisfying

$$(8.1) \quad \phi(\iota^B(c)) = \iota^D(c) \quad \text{and} \quad \delta_D(\phi(b))(1_D \otimes a) = (\phi \otimes \text{id}_A)(\delta_B(b)(1_B \otimes a))$$

for all $c \in C$, $b \in B$ and $a \in A$. Evidently, all dilations of a fixed partial coaction δ_C form a category; we denote this category by $\mathfrak{Dil}(\delta_C)$.

8.9. Remark. The second equation in (8.1) is equivalent to the condition that ϕ is a morphism of left A' -modules, that is, $\omega \triangleright \phi(b) = \phi(\omega \triangleright b)$ for all $b \in B$ and $\omega \in A'$.

If δ_C is injective and regular, then the dilation $\mathfrak{G}(\delta_C)$ is terminal among the minimal ones:

8.10. Proposition. Let δ_C be an injective, regular partial coaction of a C^* -bialgebra (A, Δ) on a C^* -algebra C , let $\mathcal{B} = (B, \delta_B, \iota)$ be a minimal dilation of δ_C , and suppose that A has the slice map property. Then there exists a unique morphism $\phi_{\mathcal{B}}$ from \mathcal{B} to $\mathfrak{G}(\delta_C)$, and on the level of C^* -algebras, $\phi_{\mathcal{B}}$ is surjective. For each $b \in B$, the image $\phi_{\mathcal{B}}(b)$ is the restriction of $\delta_B(b)$ to the ideal $\iota(C) \otimes A \cong C \otimes A$ in $B \otimes A$.

Proof. Uniqueness follows from the fact that B is generated by $\iota(C)$ and $A' \triangleright \iota(C)$.

To prove existence, define $\phi_{\mathcal{B}}$ as in (3). Since ι is a weak morphism, $\phi_{\mathcal{B}} \circ \iota = \delta_C$. The relation $(\text{id}_B \otimes \Delta)\delta_B = (\delta_B \otimes \text{id}_A)\delta_B$ implies that

$$(\phi_{\mathcal{B}} \otimes \text{id}_A)(\delta_B(b)(b' \otimes a)) = (\text{id}_C \otimes \Delta)(\phi_{\mathcal{B}}(b))(\phi_{\mathcal{B}}(b') \otimes a)$$

for all $b, b' \in B$ and $a \in A$; in particular,

$$\phi_{\mathcal{B}}(\omega \triangleright \iota(c))\phi_{\mathcal{B}}(b) = (\text{id}_C \otimes \text{id}_A \otimes \omega)((\text{id}_C \otimes \Delta)\delta_C(C))\phi_{\mathcal{B}}(b).$$

for all $c \in C$ and $\omega \in C'$. Now, the definition of $\mathfrak{G}(C)$ and minimality of \mathcal{B} imply $\phi_{\mathcal{B}}(B) = \mathfrak{G}(C)$. \square

If (A, Δ) is a C^* -quantum group, then the morphism above is injective and hence an isomorphism. To show this, we use the following observation:

8.11. Lemma. Let δ_B be a coaction of a C^* -quantum group (A, Δ) on a C^* -algebra B and let $b, b' \in M(B)$. Then $\delta_B(b)(b' \otimes 1_A) = 0$ if and only if $(b \otimes 1_A)\delta_B(b') = 0$.

Proof. Choose a modular multiplicative unitary W for (A, Δ) so that $\Delta(a) = W(a \otimes 1)W^*$ for all $a \in A$. Then

$$\begin{aligned} (\delta_B \otimes \text{id}_A)(\delta_B(b)) \cdot (\delta_B(b') \otimes 1_A) &= (\text{id}_B \otimes \Delta)(\delta_B(b)) \cdot (\delta_B(b') \otimes 1_A) \\ &= W_{23}(\delta_B(b) \otimes 1_A)W_{23}^*(\delta_B(b') \otimes 1_A). \end{aligned}$$

Since $\delta_B \otimes \text{id}_A$ is injective and W is unitary, we can conclude that $\delta_B(b)(b' \otimes 1_A) = 0$ if

$$(8.2) \quad (\delta_B(b) \otimes 1_A)W_{23}^*(\delta_B(b') \otimes 1_A) = 0.$$

A similar argument shows that $(b \otimes 1_A)\delta_B(b') = 0$ if and only if

$$(8.3) \quad (\delta_B(b) \otimes 1_A)W_{23}(\delta_B(b') \otimes 1_A) = 0.$$

Now, both (8.2) and (8.3) are equivalent to the condition $\delta_B(b)(1_B \otimes \hat{A})\delta_B(b') = 0$. \square

We can now prove claim (2) stated in the introduction to this section:

8.12. Proposition. *Let δ_C be an injective, regular partial coaction of a C^* -quantum group (A, Δ) on a C^* -algebra C , suppose that A has the slice map property, and let \mathcal{B} be a minimal dilation of δ_C . Then the morphism $\phi_{\mathcal{B}}$ from \mathcal{B} to $\mathfrak{G}(\delta_C)$ is an isomorphism.*

Proof. Write $\mathcal{B} = (B, \delta_B, \iota)$. It suffices to show that $\phi_{\mathcal{B}}$ is injective on the level of C^* -algebras. On the direct summand $\iota_C(C) \subseteq B$, the morphism $\phi_{\mathcal{B}}$ is given by $\iota_C(c) \mapsto \delta_C(c)$ and hence injective. Since B is minimal, the direct summand $(1_B - \iota(1_C))B$ of B is generated by $(1_B - \iota(1_C))(A' \triangleright \iota(C))$. Given a non-zero $b \in (1_B - \iota(1_C))B$, we therefore find some $c \in C$ such that $\delta_B(\iota(c))(b \otimes 1_A)$ is non-zero, and then $(\iota(c) \otimes 1_A)\delta_B(b)$ is non-zero by the lemma above, whence $\phi_{\mathcal{B}}(b)$ is non-zero. \square

8.13. Corollary. *Let (A, Δ) be a C^* -quantum group and suppose that A has the slice map property. Then all minimal dilations of an injective, regular partial coaction of (A, Δ) are isomorphic.*

We can now summarize the results of this section in categorical language [20].

8.14. Theorem. *Let (A, Δ) be a C^* -quantum group, where A has the slice map property, and let δ_C be an injective, regular partial coaction of (A, Δ) . Then $\mathfrak{Dil}(\delta_C)$ has initial objects and these are precisely the minimal dilations of δ_C .*

Proof. Let \mathcal{B} be a dilation of δ_C . By Proposition 8.4, \mathcal{B} contains a minimal dilation \mathcal{B}_0 which comes with an obvious morphism to \mathcal{B} , and is isomorphic to $\mathfrak{G}(\delta_C)$ by Proposition 8.12. Hence, we get a morphism from $\mathfrak{G}(\delta_C)$ to \mathcal{B} , and this morphism is necessarily unique because $\mathfrak{G}(\delta_C)$ is minimal. Now, Corollary 8.13 completes the proof. \square

Let (A, Δ) be a C^* -quantum group. Denote by $\mathfrak{P}\mathfrak{Coact}(A, \Delta)$ the category of all injective, regular partial coactions of (A, Δ) , where the morphisms between two partial coactions δ_B and δ_D on C^* -algebras B and D , respectively, are all injective $*$ -homomorphisms $\phi: B \rightarrow D$ that are weak morphisms in the sense of Definition 6.7. Denote by $\mathfrak{Coact}(A, \Delta)$ the category of all injective coactions of (A, Δ) , where the morphisms between two coactions δ_B and δ_D on C^* -algebras B and D , respectively, are all injective $*$ -homomorphisms $\phi: B \rightarrow D$ satisfying $\omega \triangleright \phi(b) = \phi(\omega \triangleright b)$ for all $\omega \in A'$ and $b \in B$.

8.15. Theorem. *Let (A, Δ) be a C^* -quantum group and suppose that A has the slice map property. Then the obvious forgetful functor $\mathfrak{U}: \mathbf{Coact}(A, \Delta) \rightarrow \mathbf{PCoact}(A, \Delta)$ has a left adjoint \mathfrak{G} which sends a partial coaction δ_C on a C^* -algebra C to the restriction of $\text{id}_C \otimes \Delta$ to $\mathfrak{G}(C)$. The unit η of this adjunction is given by $\eta_{\delta_C} = \delta_C$, and the counit ε of this adjunction is the identity.*

Proof. For every partial coaction δ_C in $\mathbf{PCoact}(A, \Delta)$, the comma category $\delta_C \rightarrow \mathfrak{U}$ has an initial object by the preceding theorem. The assertions on the unit and counit of the adjunction are easily verified. \square

We call the functor \mathfrak{G} obtained above the *globalization functor*.

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